Original Research Article

A New Inequality with Its Application in Solving a Problem of Inequalities

ABSTRACT

This article first puts forwards and proves a new inequality, then use the inequality to solve a problem related with a series of inequalities. Detail mathematical reasoning and proofs are presented. The results are valuable to learn and research inequalities.

Keywords: inequality, sort, mathematical reasoning, ascending order

1. INTRODUCTION

Recent research about distribution of the positive integers in T_3 tree, as introduced in ([1],[2],[3],[4] and [5]), has come across a problem of putting a sequence of numbers in their ascending order. The numbers are $\frac{2}{y+1}$, $\frac{2}{\alpha+1}$, $\sqrt{\frac{1}{y}}$, $\frac{2}{x+1}$, $\sqrt{\frac{1}{\alpha}}$, $\sqrt{\frac{1}{x}}$, $\frac{2\alpha}{\alpha+1}$, $\sqrt{x}$, $\frac{2y}{y+1}$, $\frac{x+1}{2}$, $\sqrt{\alpha}$ and $\sqrt{y}$. The problem involves in proving a series of the inequalities. Look into the reference handbooks ([6],[7] and [8]), no referable references were found. Thereby this paper investigates the problem and finds out a solution.

2. PRELIMINARIES

2.1 Symbols and Notations

Symbol $A \otimes B$ means A holds and simultaneously B holds. Symbol $A \Rightarrow B$ means conclusion B can be derived from condition A.

2.2 Lemma 1 (See in [9]). Let α be a real number with $\alpha \in (1,4) \cup (4,\infty)$;then

$$f(\alpha) = \frac{\sqrt{\alpha}}{2-\sqrt{\alpha}} - \left(\frac{\alpha+1}{2}\right)^2 .$$

thus

$$0 < f(\alpha) < \infty,\ \alpha \in (1,4)$$

and

$$-\infty < f(\alpha) < (-17.345),\ \alpha \in (4,6)\ .$$

3. MAIN RESULT AND PROOF

Theorem1. Let α be a real number with $a \in (1,4)$; then

$$f(\alpha) = 2\sqrt{\alpha} - 1 - (\frac{2\alpha}{\alpha+1})^2$$

thus

$$0 < f(\alpha) < 0.44,\ \alpha \in (1,4)\ .$$

Proof. Let $f(\alpha) = 2\sqrt{\alpha} - 1 - (\frac{2\alpha}{\alpha+1})^2$, $\alpha \in (1,4)$.

Simplify the function

$$f(\alpha) = \frac{(2\sqrt{\alpha}-1)(\alpha+1)^2 - 4\alpha^2}{(\alpha+1)^2}\ .$$

The denominator is greater than zero obviously.

Assume that α+1 = t; then $2 < t < 5$,thus

$$f(\alpha) = (2\sqrt{t-1} - 1) \times t^2 - 4(t-1)^2\ .$$

Let $h(t) = (2\sqrt{t-1} - 1) \times t^2 - 4(t-1)^2$; then

$$h^{'}(t) = \frac{4t(t+1) + t^2}{\sqrt{t-1}} - 10t + 8\ .$$

When $t \in (2,5)$, it obviously holds

$$12 < \frac{4t(t+1) + t^2}{\sqrt{t-1}} < 52.5$$

and

$$12 < 10t - 8 < 42\ .$$

Direct calculation yields

$$0 < h^{'}(t) < 10.5\ .$$

This means $h(t)$ is monotonically increasing in the condition of $t \in (2,5)$. Under the condition $h(2) = 0$ and $0 < h^{'}(t) < 10.5$ when $t \in (2,5)$, it is obtained $h(t) > 0$. Meanwhile, this conclusion illustrates $f(\alpha) > 0$ if $\alpha \in (1,4)$.

Theorem2. Let $1 < \alpha < 4$, x and y satisfy

$$1 < 2\sqrt{\alpha} - 1 \le x \le (\frac{2y}{y+1})^2 \le \alpha \le y \le (\frac{\alpha+1}{2})^2 < 4 \qquad (1)$$

then

$$\frac{2}{w+1} \le \frac{2}{y+1} \le \frac{2}{\alpha+1} \le \sqrt{\frac{1}{y}} \le \frac{2}{x+1} \le \sqrt{\frac{1}{\alpha}} \le \sqrt{\frac{1}{x}} \le 1 \le \frac{2\alpha}{\alpha+1} \le \sqrt{x} \le \frac{2y}{y+1} \le \sqrt{\alpha} \le \frac{x+1}{2} \le \sqrt{y} \le \sqrt{w}$$

Proof. (1) $u \le x \le \alpha \le y \le w \Rightarrow \sqrt{u} \le \sqrt{x} \le \sqrt{\alpha} \le \sqrt{y} \le \sqrt{w}$.

(2) $x \ge 2\sqrt{\alpha} - 1 \Rightarrow \frac{x+1}{2} \ge \sqrt{\alpha}$.

(3) $\frac{(\frac{2\alpha}{\alpha+1})^2}{\alpha} = \frac{4\alpha}{(\alpha+1)^2} \le \frac{4\alpha}{4\alpha} = 1 \Rightarrow \frac{2\alpha}{\alpha+1} \le \sqrt{\alpha}$.

(4) $\frac{y}{y+1}-\frac{\alpha}{\alpha+1}=\frac{y(\alpha+1)-\alpha(y+1)}{(y+1)(\alpha+1)}=\frac{y-\alpha}{(y+1)(\alpha+1)}\geq 0\Rightarrow\frac{2y}{y+1}\geq\frac{2\alpha}{\alpha+1}$.

(5) $\frac{x+1}{2}-\frac{2y}{y+1}=\frac{(x+1)(y+1)-2y}{2(y+1)}=\frac{(x-1)y+x+1}{2(y+1)}\geq 0\Rightarrow\frac{x+1}{2}\geq\frac{2y}{y+1}$.

(6) By Lemma 1, $f(\alpha)=\frac{\sqrt{\alpha}}{2-\sqrt{\alpha}}-(\frac{\alpha+1}{2})^2>0,\alpha\in(1,4)$, it immediately leads to

$$\frac{\sqrt{\alpha}}{2-\sqrt{\alpha}}\geq(\frac{\alpha+1}{2})^2\geq y\Rightarrow 2y-y\sqrt{\alpha}\leq\sqrt{\alpha}\Rightarrow\frac{2y}{y+1}\leq\sqrt{\alpha}.$$

(7) By Theorem 1, $f(\alpha)=2\sqrt{\alpha}-1-(\frac{2\alpha}{\alpha+1})^2>0,\alpha\in(1,4)$. $2\sqrt{\alpha}-1\geq(\frac{2\alpha}{\alpha+1})^2$ and $x>2\sqrt{\alpha}-1$ result in

$$\sqrt{x}\geq\frac{2\alpha}{\alpha+1}.$$

(8) By Theorem 1, it holds

$$2\sqrt{y}-1>(\frac{2y}{y+1})^2, y\in(1,4).$$

Therefore, $x\leq(\frac{2y}{y+1})^2\otimes(\frac{2y}{y+1})^2\leq 2\sqrt{y}-1\Rightarrow\frac{x+1}{2}\leq\sqrt{y}$.

(9) $\sqrt{x}\leq\frac{2y}{y+1}$ is from given condition.

(10) $w\geq y\geq\alpha\geq x\Rightarrow y+1\geq\alpha+1\geq x+1\Rightarrow\frac{2}{x+1}\geq\frac{2}{\alpha+1}\geq\frac{2}{y+1}\geq\frac{2}{w+1}$.

(11) $y\geq\alpha\geq x\Rightarrow\sqrt{\frac{1}{x}}\geq\sqrt{\frac{1}{\alpha}}\geq\sqrt{\frac{1}{y}}$.

(12) $\frac{x+1}{2}\geq\sqrt{\alpha}\Rightarrow\frac{1}{x+2}\leq\sqrt{\frac{1}{\alpha}}$.

(13) $y\leq(\frac{\alpha+1}{2})^2\Rightarrow\sqrt{y}\leq\frac{\alpha+1}{2}\Rightarrow\frac{2}{\alpha+1}\leq\sqrt{\frac{1}{y}}$.

(14) $x\leq(\frac{2y}{y+1})^2\leq 2\sqrt{y}-1\Rightarrow\frac{x+1}{2}\leq\sqrt{y}\Rightarrow\sqrt{\frac{1}{y}}\leq\frac{2}{x+1}$ (the proof is similar to (8)).

Accordingly, it holds

$$\frac{2}{y+1}\leq\frac{2}{\alpha+1}\leq\sqrt{\frac{1}{y}}\leq\frac{2}{x+1}\leq\sqrt{\frac{1}{\alpha}}\leq\sqrt{\frac{1}{x}}\leq 1\leq\frac{2\alpha}{\alpha+1}\leq\sqrt{x}\leq\frac{2y}{y+1}\leq\sqrt{\alpha}\leq\frac{x+1}{2}\leq\sqrt{y}.$$

Figure 1 shows the detailed arrangement about members of this inequation.

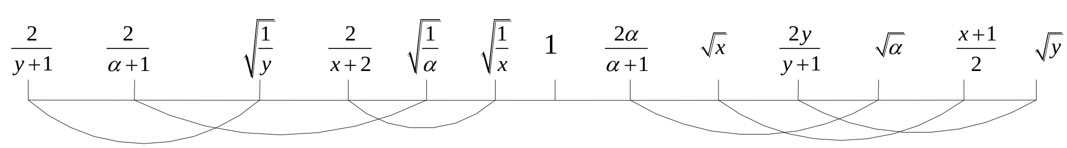

Fig. 1. Geometrical description of the inequation

3. CONCLUSION AND FUTURE WORK

This paper solves the problem that came across during the study of distribution of integers in T_3 tree, testifies a series of inequations and provides process of proofs for it. Solutions of this paper is useful to compare inequation and investigate the integer's location in T_3 tree. In the end, there is one amusing problem with the solved problem above, that is changing the condition (1) into the following one

$$1<(\frac{u+1}{2})^2 \le 2\sqrt{\alpha}-1 \le x \le (\frac{2y}{y+1})^2 \le \alpha \le y \le (\frac{\alpha+1}{2})^2 \le w < 4 .$$

Yields a more complicated distribution about u,x,y,α and w . Furthermore, it has been no what the distribution is if the condition is changed to be

$$1<(\frac{u+1}{2})^2 \le 2\sqrt{\alpha}-1 \le x \le \beta \le (\frac{2y}{y+1})^2 \le \chi \le \alpha \le y \le (\frac{\alpha+1}{2})^2 \le w < 4 .$$

Hope readers to join and to solve the problems.

COMPETING INTERESTS DISCLAIMER:

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REFERENCES

[1] Wang X, Jin Y. More Connections on Valuated Binary Tree and Their Applications in Factoring Odd Integers[J]. Asian Research Journal of Mathematics, 2021, 17(6):13-26.
DOI: 10.9734/ARJOM/2021/v17i530297.

[2] Wang X, Luo J. Connections on Valuated Binary Tree and Their Applications in Factoring Odd Integers. Asian Research Journal of Mathematics, 2021, 17(3):134-153.
DOI: 10.9734/ARJOM/2021/v17i630305.

[3] Wang X, Guo H. Distribution of RSA Number's Divisor on T3 Tree. International Journal of Information and Electronics Engineering, 2019, 9(1):23-29.
DOI: 10.18178/IJIEE.2019.9.1.699.

[4] Wang X, Guo H. Some Divisibility Traits on Valuated Binary Trees. International Journal of Information and Electronics Engineering, 2019,9(1):1-11.
DOI:10.17706/ijapm.2019.9.1.1-11.

[5] Wang X. Valuated Binary Tree: A New Approach in Study of Integers. International Journal of Scientific and Innovative Mathematical Research, 2016.
DOI:10.20431/2347-3142.0403008

[6] Kuang J. Applied Inequalities. 4th ed. Shandong Science and technology Press; 2010.

[7] Pham KH. Secrets in Inequalities (volume I). Harbin Institute of Technology Press; 2010.

[8] Pachpatte BG. Mathematical inequalities, North-Holland mathematical library; 2005.

[9] Han MX, Yan JH. Bound Estimation of Some Special Functions. Asian Research Journal of Mathematics,2021:(in press).