Solving generalized nonlinear Schrödinger equation by Adomian decomposition technique

Abstract

In this paper, using a suitable change of variable and applying the Adomian decomposition method to the generalized nonlinear Schrödinger equation, we obtain the analytical solution, taking into account the parameters such as the self-steepening factor, the second order dispersive parameter, the third order dispersive parameter and the nonlinear Kerr effect coefficient, for pulses that contain just a few optical cycle. The analytical results are performed numerically. Under influence of these effects, pulse did not maintain its initial shape.

Keywords:Adomian method, nonlinear Schrödinger equation, ultrashort pulse propagation.
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1 Introduction

The propagation process of ultrashort laser pulses in a nonlinear medium is nowadays in the center of interest because of its important application in the optical telecommunication. Important effects involved in this propagation process have been theoretically and experimentally considered by several authors [1]. In this paper, we solve the generalized nonlinear Schrödinger (GNLSE) by using Adomian decomposition technique. In this equation, the parameters such as the self-steepening factor, the second order dispersive parameter, the third order dispersive parameter and the nonlinear Kerr effect coefficient and the self-frequency shift are included. In a series of remarquable papers [1-3], the authors have studied this equation using various approaches. One of the methods is the Adomian decomposition method for solving a wide range of physical problems [4-10]. Several modifications were improved its ability in [11-19]. An advantage of this method is that it can provide analytical approximation or an approximated solution to a wide class of nonlinear equations without linearization, perturbation, closure approximation or discretization methods. Its abilities attracted many authors to use this method for solving physical problems. Our paper is organized as follows: in section II, we present the analytical solutions. Section III contains the results and discussion and section IV contains the conclusions.

2 Analytical results

The generalized nonlinear Schrödinger equation deals with the pulse envelope $A(z,t)$ related to the electric field $E(z,t)$. The evolution of $A(z,t)$ inside the dispersive nonlinear medium is then governed by the standard NLS equation generalized thanks to the additional terms that represent the higher-order nonlinear and dispersion effects. Such a generalized nonlinear Schrödinger equation has the form [20]

$$i\frac{\partial A}{\partial \xi} + \frac{1}{2}\frac{\partial^2 A}{\partial \tau^2} + i\delta_3\frac{\partial^3 A}{\partial \tau^3} + \beta_0 n_2 L_D\left[|A|^2 A + is\frac{\partial}{\partial \tau}\left(|A|^2 A - \tau_R A\frac{\partial |A|^2}{\partial \tau}\right)\right] = 0$$

$$A(0,\tau) = \sqrt{I_0} f_p(\tau)$$

where $\xi = z/L_D$ is the distance normalized to the dispersion length and $\tau = (t - z/v_g)/T_0$ is the time normalized to input pulse width T_0, $\delta_3 = \beta_3/(6T_0|\beta_2|)$ takes into account the third-order dispersion effects governed by β_3, $s = 1/(\omega_0 T_0)$ is the parameter responsible for self-steepening and $f_p(\tau)$ governs the pulse shape. Throughout this paper, intrapulse Raman scattering τ_R is equal to zero.

Now we provide the analytical solution of the generalized nonlinear Schrödinger equation. Setting

$$\kappa = \beta_0 n_2 L_D, \quad \eta = \varepsilon\xi + \tau, \quad \eta_{\xi=0} = \tau, \quad g(\eta) = A(\xi,\tau)$$

for small positive number ε, the above partial differential equation reduces to the following nonlinear functional equation

$$\begin{aligned} g(\eta) - \Bigg\{ & \frac{i}{2\delta_3}\int_{\eta_0}^{\eta} g(\xi)d\xi - \frac{\varepsilon}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi} g(\sigma)d\sigma d\xi \\ & - \frac{s\kappa}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi} |g(\sigma)|^2 |g(\sigma)| d\sigma d\xi + \frac{i\kappa}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi}\int_{\sigma_0}^{\sigma} |g(y)|^2 g(y) dy d\sigma d\xi \Bigg\} \\ & = g(\eta_0) + \left[g'(\eta_0) - \frac{i}{2\delta_3}g(\eta_0)\right](\eta - \eta_0) \\ & + \frac{1}{2}\left[g''(\eta_0) - \frac{i}{2\delta_3}g'(\eta_0) + \frac{\varepsilon}{\delta_3}g(\eta_0) + \frac{s\kappa}{\delta_3}|g(\eta_0)|^2 g(\eta_0)\right](\eta - \eta_0)^2 \end{aligned} \tag{2.1}$$

equivalently

$$g - N(g) = q \tag{2.2}$$

where

$$N(g) = \frac{i}{2\delta_3}\int_{\eta_0}^{\eta} g(\xi)d\xi - \frac{\varepsilon}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi} g(\sigma)d\sigma d\xi \tag{2.3}$$

$$- \frac{s\kappa}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi} |g(\sigma)|^2|g(\sigma)|d\sigma d\xi + \frac{i\kappa}{\delta_3}\int_{\eta_0}^{\eta}\int_{\xi_0}^{\xi}\int_{\sigma_0}^{\sigma} |g(y)|^2 g(y)dyd\sigma d\xi$$

$$q(\eta) = g(\eta_0) + \Big[g'(\eta_0) - \frac{i}{2\delta_3}g(\eta_0)\Big](\eta - \eta_0)$$

$$+ \frac{1}{2}\Big[g''(\eta_0) - \frac{i}{2\delta_3}g'(\eta_0) + \frac{\varepsilon}{\delta_3}g(\eta_0) + \frac{s\kappa}{\delta_3}|g(\eta_0)|^2 g(\eta_0)\Big](\eta - \eta_0)^2$$

In order to obtain the analytical solution of the equation (2.2), the Adomian method is used for solving nonlinear functional equations such as the equation (2.2), where N is a nonlinear operator from a Hilbert space H into H, q is a given function in H. We are looking for f satisfying (2.2) and we assume that (2.2) has a unique solution for every $q \in H$.
The Adomian method consists in representing g as follows [7-13]

$$g = \sum_{n=0}^{+\infty} g_n \tag{2.4}$$

The nonlinear operator N is decomposed as follows

$$N(g) = \sum_{n=0}^{+\infty} A_n \tag{2.5}$$

where the A_n are functions (Adomian's polynomials) of $g_0, \dots, g_n$ that are obtained by writing:

$$p = \sum_{n=0}^{+\infty} \lambda^n g_n, \;\; N\Big(\sum_{n=0}^{+\infty} \lambda^n g_n\Big) = \sum_{n=0}^{+\infty} \lambda^n A_n \tag{2.6}$$

where λ is a parameter introduced for convenience. From (2.6) we deduced the A_n values by the formulae

$$n!A_n = \frac{d^n}{d\lambda^n}\left[N\Big(\sum_{i=0}^{+\infty} \lambda^i g_i\Big)\right]_{\lambda=0}, \;\; n = 0, 1, 2, 3, \dots \tag{2.7}$$

Thus we will compute in a recurrent manner g_n and A_n to the following relationships:

$$\begin{cases} g_0 = q \\ g_1 = A_0 \\ \vdots \\ g_n = A_{n-1} \\ \vdots \end{cases}$$

Indeed for simplicity reasons we choose $g'(\eta_0) = ag(\eta_0), \;\; g''(\eta_0) = bg(\eta_0)$ for real numbers a, b and we take into account $g(\eta_0) = A(0, \tau)$, the initial pulse; then we have $g_0, \; g_1$ and the 2-term approximation of $g = \sum_{n\geq 0} g_n$ such as:

$$g_0(\eta) = e^{-\tau^2/2}\left[1 + \left(a - i\frac{1}{2\delta_3}\right)(\eta - \eta_0) + \frac{1}{2}\left(b - i\frac{a}{2\delta_3} + \frac{\varepsilon}{\delta_3} + \frac{s\kappa}{\delta_3}e^{-\tau^2}\right)(\eta - \eta_0)^2\right] \tag{2.8}$$

$$g_1(\eta) = N(g_0) \tag{2.9}$$

$$
\begin{aligned}
g_1(\eta) = e^{-\tau^2/2}\Bigg\{ & \frac{i}{2\delta_3}\Bigg[\eta-\eta_0+\frac{1}{2}\Big(a-i\frac{1}{2\delta_3}\Big)(\eta-\eta_0)^2 + +\frac{1}{6}\Big(b-i\frac{a}{2\delta_3}+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)(\eta-\eta_0)^3\Bigg] \\
& -\frac{\varepsilon}{\delta_3}\Bigg[\frac{1}{2}(\eta-\eta_0)^2+\frac{1}{6}\Big(a-i\frac{1}{2\delta_3}\Big)(\eta-\eta_0)^3 + +\frac{1}{24}\Big(b-i\frac{a}{2\delta_3}+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)(\eta-\eta_0)^4\Bigg] \\
& -\frac{s\kappa}{\delta_3}e^{-\tau^2}\Bigg[\frac{1}{2}(\eta-\eta_0)^2+\frac{1}{6}(3a-\frac{1}{2\delta_3}i)(\eta-\eta_0)^3 + +\frac{1}{12}(3a^2+\frac{3}{2}b+\frac{3\varepsilon}{2\beta_3}+\frac{1}{4\delta_3^2}+\frac{3s\kappa}{2\delta_3}e^{-\tau^2}-\frac{5a}{4\delta_3}i)(\eta-\eta_0)^4 \\
& +\frac{1}{20}\Big[a^3+3ab+\frac{3a\varepsilon}{\delta_3}+\frac{a}{2\delta_3^2}+\frac{3as\kappa}{\delta_3}e^{-\tau^2}-\frac{i}{2\delta_3}\Big(2a^2+b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big](\eta-\eta_0)^5 \\
& +\frac{1}{30}\Big[\frac{1}{2}\Big(a^2+b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)+a^2\Big(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)+ \\
& +\frac{a^2}{16\delta_3^2}+\frac{1}{4}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)^2-\frac{a}{4\delta_3}i\Big(a^2+3b+\frac{3\varepsilon}{\delta_3}+\frac{3}{4\delta_3^2}+\frac{3s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big](\eta-\eta_0)^6+ \\
& +\frac{1}{42}\Big[\frac{a}{2}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big) + +a\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big) \\
& -\frac{a^2}{4\delta_3}i(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2})-\frac{i}{2\beta_3}\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big)\Big](\eta-\eta_0)^7 \\
& +\frac{1}{56}\Big[\frac{1}{2}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big)-\frac{a}{4\delta_3}i\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big)\Big](\eta-\eta_0)^8\Bigg] \\
& +\frac{i\kappa}{\delta_3}e^{-\tau^2}\Bigg[\frac{1}{6}(\eta-\eta_0)^3+\frac{1}{24}(3a-\frac{1}{2\delta_3}i)(\eta-\eta_0)^4+\frac{1}{60}(3a^2+\frac{3}{2}b+\frac{3\varepsilon}{2\delta_3}+\frac{1}{4\delta_3^2}+\frac{3s\kappa}{2\delta_3}e^{-\tau^2}-\frac{5a}{4\delta_3}i)(\eta-\eta_0)^5+ \\
& +\frac{1}{120}\Big[a^3+3ab+\frac{3a\varepsilon}{\delta_3}+\frac{a}{2\delta_3^2}+\frac{3as\kappa}{\delta_3}e^{-\tau^2}-\frac{i}{2\delta_3}\Big(2a^2+b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big](\eta-\eta_0)^6 \\
& +\frac{1}{210}\Big[\frac{1}{2}\Big(a^2+b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)+a^2\Big(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big) \\
& +\frac{a^2}{16\delta_3^2}+\frac{1}{4}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)^2-\frac{a}{4\delta_3}i\Big(a^2+3b+\frac{3\varepsilon}{\delta_3}+\frac{3}{4\delta_3^2}+\frac{3s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big](\eta-\eta_0)^7 \\
& +\frac{1}{336}\Big[\frac{a}{2}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)+a\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big) \\
& -\frac{a^2}{4\delta_3}i(b+\frac{\varepsilon}{\delta_3}+\frac{1}{4\delta_3^2}+\frac{s\kappa}{\delta_3}e^{-\tau^2})-\frac{i}{2\delta_3}\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big)\Big](\eta-\eta_0)^8+ \\
& \frac{1}{504}\Big[\frac{1}{2}\Big(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2}\Big)\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big) \\
& -\frac{a}{4\delta_3}i\Big(\frac{a^2}{16\delta_3^2}+\frac{1}{4}(b+\frac{\varepsilon}{\delta_3}+\frac{s\kappa}{\delta_3}e^{-\tau^2})^2\Big)\Big](\eta-\eta_0)^9\Bigg]\Bigg\}
\end{aligned}
$$

3 Results and discussion

First of all, let's point out that the following parameters δ, s τ_R characterize the higher-order dispersive and nonlinear terms. These parameters govern respectively the effects of third order dispersion, self-steepening and the self-shift frequency. One can see that when the width of the pulse T_0 decreases, the pulse is shorter and the nonlinear parameters increase. Consequently, when the pulse is shorter, the higher effects are more important. Then, when the time T has the values of picoseconds or larger, these parameters are very small and can be neglected, so the standard NLS equation becomes inadequate. We observe that during propagation through the waveguide, where the propagation direction is coincident with the z-axis, the intensity of the pulse $A(t, z)$ decreases due to several loss mechanisms for both gaussien and hyperbolic secant profiles. From Fig. 1 we have chosen a Gaussian pulse for "$s = 0.02$, $T_0 = 10[fs]$, $\beta_2 = -1[ps^2/km]$, $\tau_R = 0$, $a = 0.5$, $b = 0.25$, $\kappa = 0.1$, $\delta_3 = 0.02[ps^3/km]$, $\epsilon = 2.5 \times 10^{-6}$, $P_0 = 1[mW]$". Fig. 1 compares the intensity profiles at a distance $z = 10L_D$ with the input Gaussian pulse. As seen there, the shape of the pulse subsided even it keeps the same Gaussian profile. We consider now the propagation of the ultrashort pulse with the initial hyperbolic secant shape with the same parameters. Fig. 2 shows changes in the electric field with propagation. As expected, the the electric field is distorted considerably when compared to that at the input taken to be: $E_{in}(\tau) = sech(\tau)cos(\tau)$. Fig. 3 presents the evolution of the intensity of the Gaussian-pulse with "$\delta_3 = 0.03[ps^3/km]$" and Fig. 4 shows the intensity of the pulse when "$\delta_3 = 0.09[ps^3/km]$". The comparison shows that when we inrease the third order dispersive term, the pulse shape exhibit oscillations. When the propagation distance is larger the oscillation of the envelope is stronger. Let's point out that the influence of the self-steepening effect is slighly observed when its value is increased.

In the results obtained above, we may see the impact that third dispersive and nonlinear effects have on the propagation of the ultrashort pulses in its whole importance. Under the influence of these effects, the propagation of the ultrashort pulses is much more complicated than in case of short pulses.

Therefore, some higher-order effects such as third-order dispersion, self-steepening play important roles in the propagation of optical pulses. The effect of self-steepening is due to the intensity-dependent group velocity of the optical pulse, which gives the pulse a very narrow width in the course of propagation.

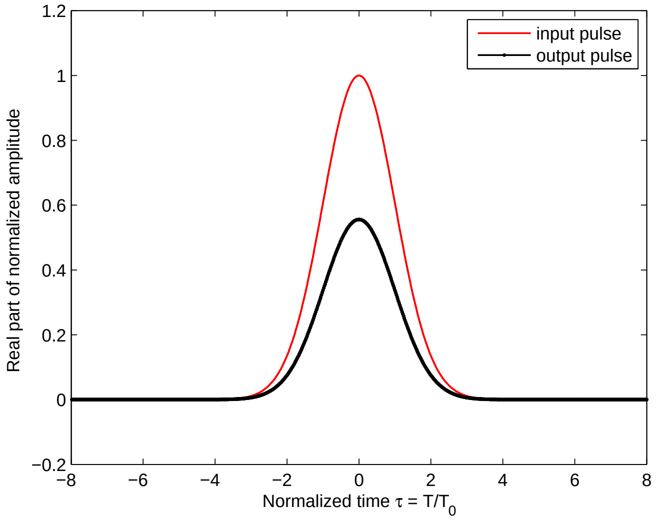

Figure 1: Gaussian-pulse envelopes as a function of propagation distance z: on top, ”z=0 [km]”; and below, the output pulse with $z = 10L_D$

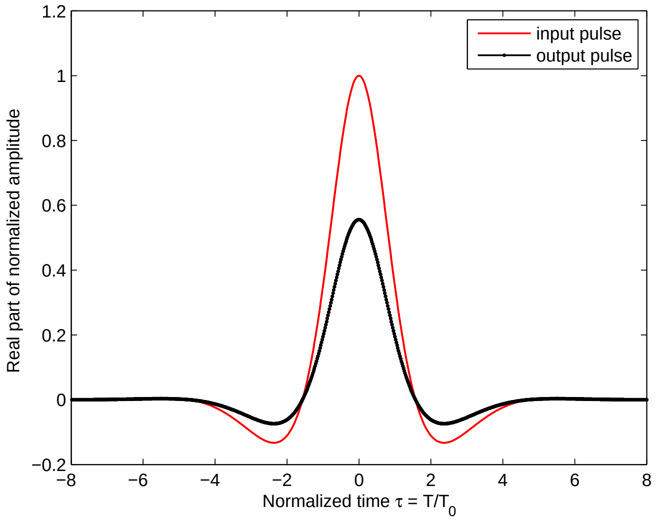

Figure 2: Hyperbolic-sech pulse envelopes as a function of propagation distance z. On top: ”z=0 [km]”; and below, the output pulse with $z = 10L_D$

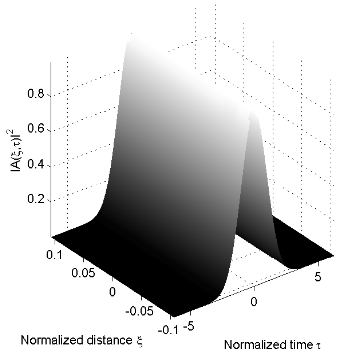

Figure 3: Evolution of the intensity of the Gaussian pulse. The parameters used are: "$\delta_3 = 0.03ps^3/km$, $s = 0.8$, $\tau_R = 0$, $T_0 = 10fs$"

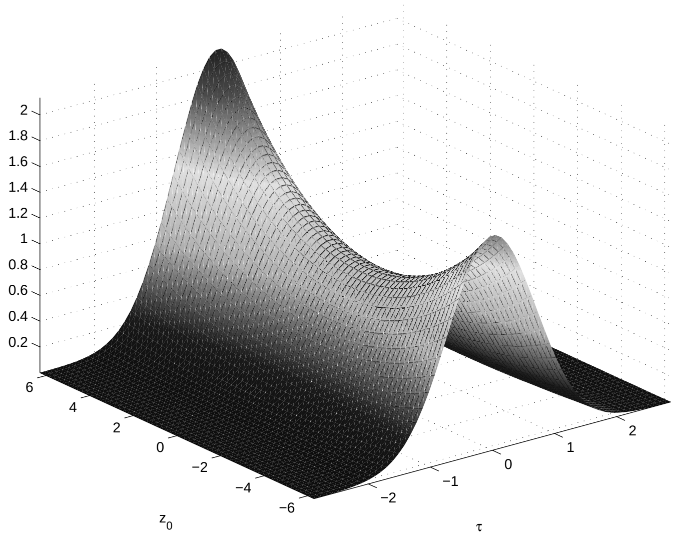

Figure 4: Evolution of the intensity of the Gaussian pulse. The parameters used are: "$\delta_3 = 0.09ps^3/km$, $s = 0.8$, $\tau_R = 0$, $T_0 = 10fs$"

4 Concluding remarks

We have applied Adomian decomposition technique to solve analytically the nonlinear Schrödinger equation for propagation of an ultrashort optical pulse inside a nonlinear medium. Our results illustrate the third order dispersive effect in pulse distorsion with an oscillatory structure. The self-steepening factor reduces the width of the pulse during propagation. The ultrashort pulses are widely used nowadays, especially in the optical telecommunication, so the results obtained in the research of these pulses are of great practical importance.

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