Impact of Measurement Error on the Power Function of Average Control Chart under Non-Normal Population

Abstract

In quality control applications measurement error often exist which results in reduced power to identify a given modification in the mean of a quality characteristic. The effect of measurement error and non-normality on the power function of the control charts for mean with control limits is studied. The non-normality is represented by the first four terms of an Edgeworth series. Tabular and visual comparison is also provided for the better comprehension.
Keywords: Inspection Error, Average Control Chart, Non-normal Population, Power Function.

1. Introduction:

Statistical process control includes a set of upcoming problems solving tools which is useful for creating stability in the process by decreasing the variability. The control chart techniques are one of the key methods in statistical process control which is most widely used in the industries. Control chart technique is mainly applied for; controlling (current process by finding and correcting problems as they occur), forecasting (the expected range of outcomes from a process), determining (whether a process is in statistical control) and analyzing (patterns of process variations). When outlining control charts, it has been a general presumption that the observations are independently and identically distributed $(i.i.d)$ under normal population. Practically, this presumption is not generally valid. Non-normality has a significant effect on the performance of the average control chart. The design consideration for an average control chart must include recognition of the degree of non-normality of the underlying data. The effect of non-normality on average control chart is discussed by Yourstone and Zimmer (2007). Chou *et al*. (2005), Singh and Singh (2014) have significantly given their contribution in the field of control chart under non-normal population. In quality control applications measurement error often exists which is known to result in reduced power to detect a given change in the mean of a quality characteristic. The effect of measurement error on control chart is considered by many authors: Bennet (1954), Linna and Woodall (2001), Duffua and Khan (2005), Chakraborty *et al.* (2017), Singh and Mishra (2017).

In this paper, we have attempted to examine the effect of inspection error on the power function of a control chart for averages where the process average can change under the non-normal population.

2. Power Function of Average Control Chart under Non-normal Population

Assuming that the true measurement *x* and the random error *e* are additive, we can write the observed measurement *X* as:

$$X = x + e, \qquad (2.1)$$

where *x* and *e* are independent.

We now assume the density function of *x* to be specified by the first four terms of the Edgeworth series as follows:

$$f(x) = \frac{1}{\sigma_p}\left\{\phi\left(\frac{x-\mu}{\sigma_p}\right) - \frac{\lambda_3}{6}\phi^{(3)}\left(\frac{x-\mu}{\sigma_p}\right) + \frac{\lambda_4}{24}\phi^{(4)}\left(\frac{x-\mu}{\sigma_p}\right) + \frac{\lambda_3^2}{72}\phi^{(6)}\left(\frac{x-\mu}{\sigma_p}\right)\right\}, \tag{2.2}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-X^2/2} \quad \text{and} \quad \phi^{(r)}(X) = \frac{d^r}{dX}\phi(\mathrm{X}).$$

The constants μ and σ_p are the mean and the standard deviation of the true quality measurement *x* and λ_3 and λ_4 are the standardized third and fourth cumulants respectively. As usual if we take $e \sim N(0, \sigma_e^2)$ and independent of *x*, the correlation coefficient ρ between the true and the observed measurement can be found out to be:

$$\rho = \frac{\sigma_p}{\sigma_X} = \frac{r}{\sqrt{1+r^2}}, \tag{2.3}$$

where σ_X is the standard deviation of *X*. We want to derive the density function of $\bar{X}$ which can be obtained as follows:

Since *x* and *e* are independent, the r^{th} cumulant of *X* is equal to the sum of the r^{th} cumulant of *x* and *e*. Further, since $e \sim N(0, \sigma_e^2)$, all the cumulants of *e* are zero except the second cumulant which is σ_e^2. Thus, if we denote the r^{th} cumulant of *x* and *X* by k_r and l_r, we have:

$$k_r = l_r\,, \qquad r \neq 2$$
$$k_2 = l_2\,. \tag{2.4}$$

Let γ_r and λ_r $(r \neq 2)$ be the r^{th} standardized cumulant of *X* and *x* respectively. Then,

$$\gamma_r = \frac{k_r}{k_2^{r/2}} = \frac{l_r}{\sigma_X^2} = \frac{l_r}{\left({}^{\sigma_p}/_{\rho}\right)^r}\,,$$

$$\text{or } \gamma_r = \rho^r \lambda_r. \tag{2.5}$$

Then the density function of *X* can be written from equation (2.2) as:

$$f(X) = \frac{1}{\sigma_X}\left\{\phi\left(\frac{X-\mu}{\sigma_X}\right) - \rho^3\frac{\lambda_3}{6}\phi^{(3)}\left(\frac{X-\mu}{\sigma_X}\right) + \rho^4\frac{\lambda_4}{24}\phi^{(4)}\left(\frac{X-\mu}{\sigma_X}\right) + \rho^6\frac{\lambda_3^2}{72}\phi^{(6)}\left(\frac{X-\mu}{\sigma_X}\right)\right\}. \tag{2.6}$$

The distribution of $\bar{X}$ for the observed samples of size *n* drawn from the population (2.6) is found by the following equation (Gayan (1952)):

$$g(\bar{X}) = \frac{\sqrt{n}}{\sigma}\left\{\phi\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) - \rho^3\frac{\lambda_3}{6\sqrt{n}}\phi^{(3)}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) + \rho^4\frac{\lambda_4}{24n}\phi^{(4)}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) + \rho^6\frac{\lambda_3^2}{72n}\phi^{(6)}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)\right\}, \tag{2.7}$$

Now integrating equation (2.7) after replacing μ by μ', we have:

$$\gamma(\bar{x}) = \left\{\Phi\left(\frac{\bar{x}-\mu'}{\sigma/\sqrt{n}}\right) - \rho^3\frac{\lambda_3}{6\sqrt{n}}\phi^{(2)}\left(\frac{\bar{x}-\mu'}{\sigma/\sqrt{n}}\right) + \rho^4\frac{\lambda_4}{24n}\phi^{(3)}\left(\frac{\bar{x}-\mu'}{\sigma/\sqrt{n}}\right) + \rho^6\frac{\lambda_3^2}{72n}\phi^{(5)}\left(\frac{\bar{x}-\mu'}{\sigma/\sqrt{n}}\right)\right\}, \tag{2.8}$$

where

$$r = \frac{\sigma_p}{\sigma_e} = \frac{\rho}{\sqrt{1-\rho^2}}. \tag{2.9}$$

It is assumed that the process comes from the $N\left(\mu, \sigma_p^2/n\right)$ but the process shifts, the data is assumed to come from $N(\mu, (\sigma_X^2 + \sigma_e^2)/n)$ under non-normal population and the value of $\bar{X}$ is plotted with control limits of $\left(\mu \pm 3\sigma_p^2/n\right)$, the power of detecting the change of process is given by:

$$P_{\bar{X}} = P_r\left\{\bar{X} \geq \mu + 3\,\sigma_p/\sqrt{n}\right\} + P_r\left\{\bar{X} \leq \mu - 3\,\sigma_p/\sqrt{n}\right\} \tag{2.10}$$

Standardizing the above equation (2.9),

$$Z = \frac{\bar{X}-\mu'}{\sqrt{(\sigma_X^2+\sigma_e^2)/n}} \tag{2.11}$$

The power function for normal distribution is obtained by converting equation (2.9) into the standardized form, we have:

$$P_{\bar{X}} = \left\{P_r\left(Z \geq (\mu-\mu')\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}} + 3\sqrt{\frac{\sigma_p^2}{n}}\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}}\right) + P_r\left(Z \leq (\mu-\mu')\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}} - 3\sqrt{\frac{\sigma_p^2}{n}}\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}}\right)\right\}, \tag{2.12}$$

$$P_{\bar{X}} = \left\{P_r\left(Z \geq (\mu-\mu')\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}} + 3\sqrt{\frac{\sigma_p^2}{(\sigma_X^2+\sigma_e^2)}}\right) + P_r\left(Z \leq (\mu-\mu')\sqrt{\frac{n}{(\sigma_X^2+\sigma_e^2)}} - 3\sqrt{\frac{\sigma_p^2}{(\sigma_X^2+\sigma_e^2)}}\right)\right\}, \tag{2.13}$$

$$P_{\bar{X}} = \left\{P_r\left(Z \geq \frac{-d\sqrt{n}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{r^2}}} + 3\,\frac{1}{\sqrt{\frac{1}{\rho^2}+\frac{1}{r^2}}}\right) + P_r\left(Z \leq \frac{-d\sqrt{n}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{r^2}}} - 3\,\frac{1}{\sqrt{\frac{1}{\rho^2}+\frac{1}{r^2}}}\right)\right\}, \tag{2.14}$$

$$P_{\bar{X}} = \left\{P_r\left(Z \leq \sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + P_r\left(Z \leq \sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\}, \tag{2.15}$$

$$P_{\bar{X}} = \left\{\Phi\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + \Phi\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\}, \tag{2.16}$$

where $d = \frac{\mu-\mu'}{\sigma}$, $r^2 = \frac{\sigma_p^2}{\sigma_e^2}$ and $\rho^2 = \frac{\sigma_p^2}{\sigma_X^2}$.

The Power Curve of the control chart when the underlying population is non-normal is obtained by putting above value of equation (2.16) in equation (2.8):

$$P_{\bar{X}} = \left\{\Phi\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + \Phi\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\} - \frac{\lambda_3}{6\sqrt{n}}\left\{\phi^{(2)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + \phi^{(2)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\} + \frac{\lambda_4}{24n}\left\{\phi^{(3)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + \phi^{(3)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\} + \frac{\lambda_3^2}{72n}\left\{\phi^{(5)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(d\sqrt{n}-3\right)\right) + \phi^{(5)}\left(\sqrt{\frac{\rho^2 r^2}{\rho^2+r^2}}\left(-3-d\sqrt{n}\right)\right)\right\}. \tag{2.17}$$

The values of the power curve are obtained by using the equation (2.17) is given in Table-1 and its diagrammatical representation is given in Fig-1.

3. Numerical Illustration and Conclusion:

In order to illustrate the result we study the effect of measurement error on power function of average control chart under non-normal population. The values of power function for some chosen values of d, $r = \infty, 2, 4, 6, 8$ and different combinations of non-normal parameter $(\lambda_3, \lambda_4) = (0,0), (0,0.5), (0.5,0), (0,-0.5), (-0.5,0), (0.5,0.5), (-0.5,-0.5)$ have been worked out using equation (2.17) and given in the Table-1. To give a visual comparison of power curves have been drawn in Figure-1 to 5.

Table-1

Power					
(λ_3, λ_4)	**d**	**r=∞**	**r=2**	**r=4**	**r=6**
$(\mathbf{0}, \mathbf{0})$	0	0.00270	0.05778	0.01860	0.01091
	0.3	0.01005	0.08049	0.03583	0.02498
	0.5	0.02994	0.12157	0.07054	0.05538
	0.8	0.11292	0.22307	0.17112	0.15207
	1.0	0.22245	0.31496	0.27451	0.25843
	1.3	0.46291	0.47661	0.47089	0.46851
	1.5	0.63837	0.58863	0.60941	0.61809
	1.8	0.84730	0.74158	0.78931	0.80776
	2.0	0.92951	0.82409	0.87592	0.89419
	2.5	0.99520	0.94931	0.97892	0.98602
	3.0	0.99990	0.99049	0.99819	0.99917
$(\mathbf{0}, \mathbf{0.5})$	0	0.00358	0.05831	0.01951	0.01183
	0.3	0.01127	0.08090	0.03660	0.02589
	0.5	0.03131	0.12182	0.07105	0.05608
	0.8	0.11337	0.22307	0.17110	0.15211
	1.0	0.22205	0.31489	0.27430	0.25815
	1.3	0.46275	0.47660	0.47084	0.46844
	1.5	0.63885	0.58870	0.60957	0.61832
	1.8	0.84724	0.74168	0.78944	0.80789
	2.0	0.92854	0.82411	0.87578	0.89390
	2.5	0.99434	0.94907	0.97836	0.98534
	3.0	0.99981	0.99027	0.99797	0.99899
$(\mathbf{0.5}, \mathbf{0})$	0	0.00013	0.05401	0.01432	0.00695
	0.3	0.00556	0.07744	0.03163	0.02053
	0.5	0.02354	0.11980	0.06713	0.05110
	0.8	0.10939	0.22402	0.17141	0.15160
	1.0	0.22663	0.31774	0.27844	0.26263
	1.3	0.47744	0.48094	0.47841	0.47774
	1.5	0.65133	0.59273	0.61632	0.62648
	1.8	0.84860	0.74377	0.79168	0.81000
	2.0	0.92427	0.82462	0.87505	0.89237
	2.5	0.97717	0.94733	0.97472	0.98003
	3.0	0.94176	0.98838	0.99106	0.98577
$(\mathbf{0}, \mathbf{-0.5})$	0	0.00181	0.05724	0.01769	0.00993
	0.3	0.00882	0.08007	0.03505	0.02406
	0.5	0.02856	0.12133	0.07003	0.05468
	0.8	0.11247	0.22306	0.17113	0.15203
	1.0	0.22285	0.31502	0.27472	0.25870
	1.3	0.46306	0.47662	0.47094	0.46858
	1.5	0.63789	0.58856	0.60924	0.61785
	1.8	0.84735	0.74148	0.78917	0.80763

	2.0	0.93047	0.82406	0.87606	0.89448
	2.5	0.99606	0.94954	0.97947	0.98669
	3.0	0.99998	0.99072	0.99840	0.99935
$(\mathbf{-0.5, 0})$	0	0.00542	0.06158	0.02299	0.01499
	0.3	0.01473	0.08354	0.04009	0.02952
	0.5	0.03650	0.12332	0.07395	0.05968
	0.8	0.11606	0.22204	0.17064	0.15230
	1.0	0.21738	0.31209	0.27033	0.25384
	1.3	0.44808	0.47228	0.46331	0.45919
	1.5	0.62690	0.58461	0.60280	0.61019
	1.8	0.84949	0.73964	0.78779	0.80687
	2.0	0.93601	0.82386	0.87766	0.89718
	2.5	0.98310	0.95121	0.98076	0.98644
	3.0	0.94215	0.99091	0.99271	0.98695
	0	0.00102	0.05454	0.01523	0.00793
	0.3	0.00679	0.07786	0.03240	0.02145
	0.5	0.02492	0.12005	0.06764	0.05180
	0.8	0.10984	0.22403	0.17140	0.15165
	1.0	0.22623	0.31768	0.27823	0.26236
$(\mathbf{0.5, 0.5})$	1.3	0.47728	0.48093	0.47836	0.47767
	1.5	0.65181	0.59281	0.61649	0.62671
	1.8	0.84855	0.74387	0.79182	0.81013
	2.0	0.92330	0.82465	0.87491	0.89208
	2.5	0.97631	0.94710	0.97417	0.97936
	3.0	0.94168	0.98815	0.99084	0.98559
	0	0.00453	0.06105	0.02207	0.01401
	0.3	0.013506	0.08312	0.03932	0.028601
	0.5	0.035128	0.12308	0.07344	0.05898
	0.8	0.115609	0.22203	0.17065	0.152251
	1.0	0.217772	0.31216	0.27054	0.254112
$(\mathbf{-0.5, -0.5})$	1.3	0.448232	0.47228	0.46335	0.459256
	1.5	0.626414	0.58454	0.60263	0.609957
	1.8	0.849543	0.73954	0.78765	0.806742
	2.0	0.936981	0.82384	0.87779	0.897466
	2.5	0.983959	0.95145	0.98131	0.987118
	3.0	0.942234	0.99113	0.99292	0.987127

The measurement error is dependent on the amount of variability in the error term. As soon as this amount increases, the effect of measurement error and non-normality on the power function becomes more serious.

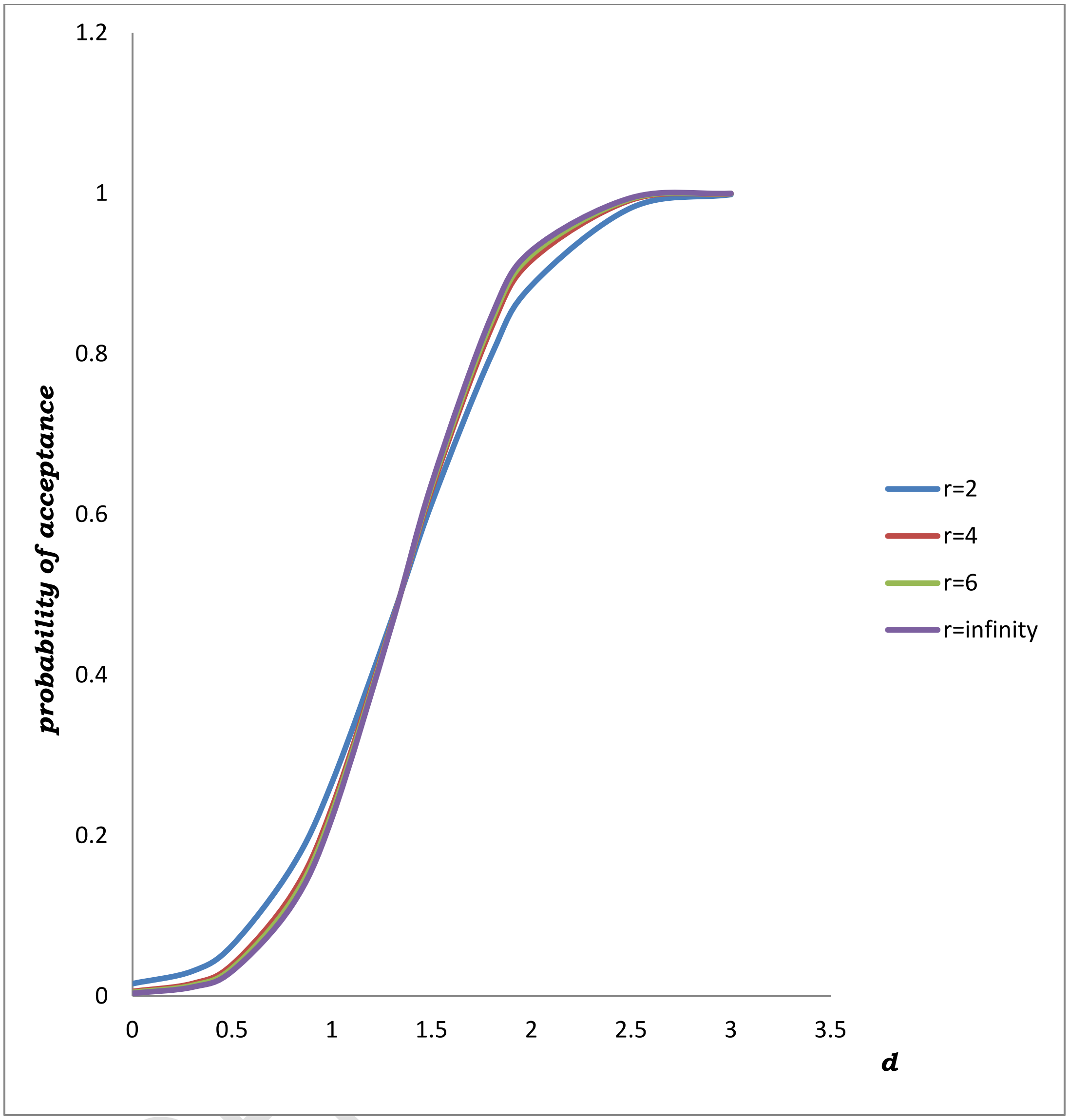

Fig-1: Power curve of average control chart when $(\lambda_3, \lambda_3) = (0,0.5)$

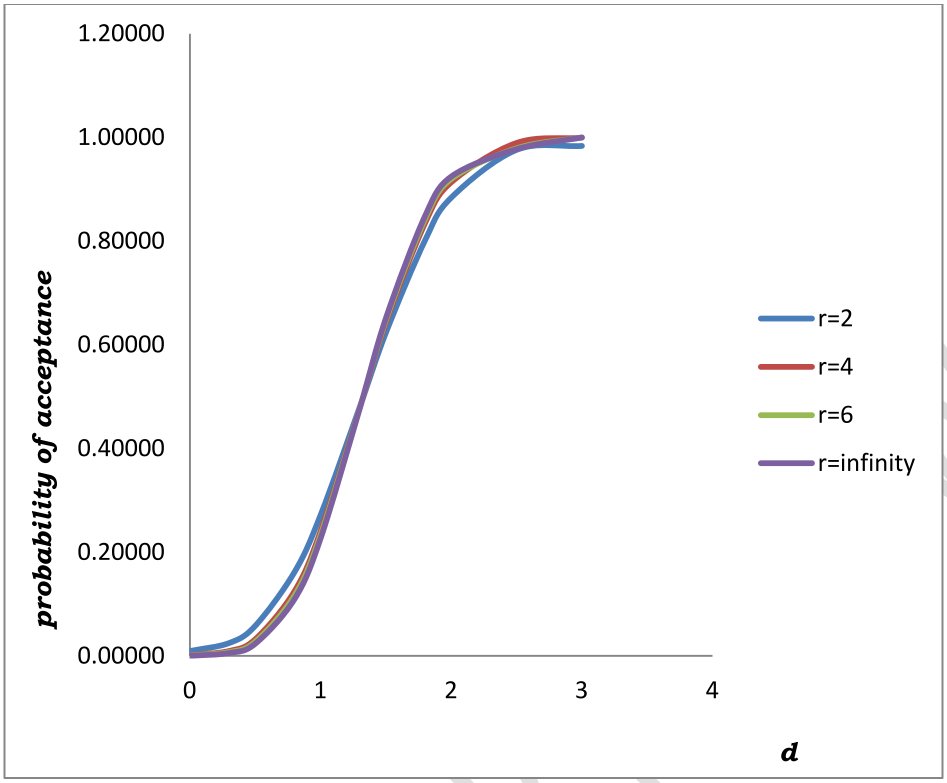

Fig-2: Power curve of average control chart when $(\lambda_3, \lambda_3) = (0.5,0)$.

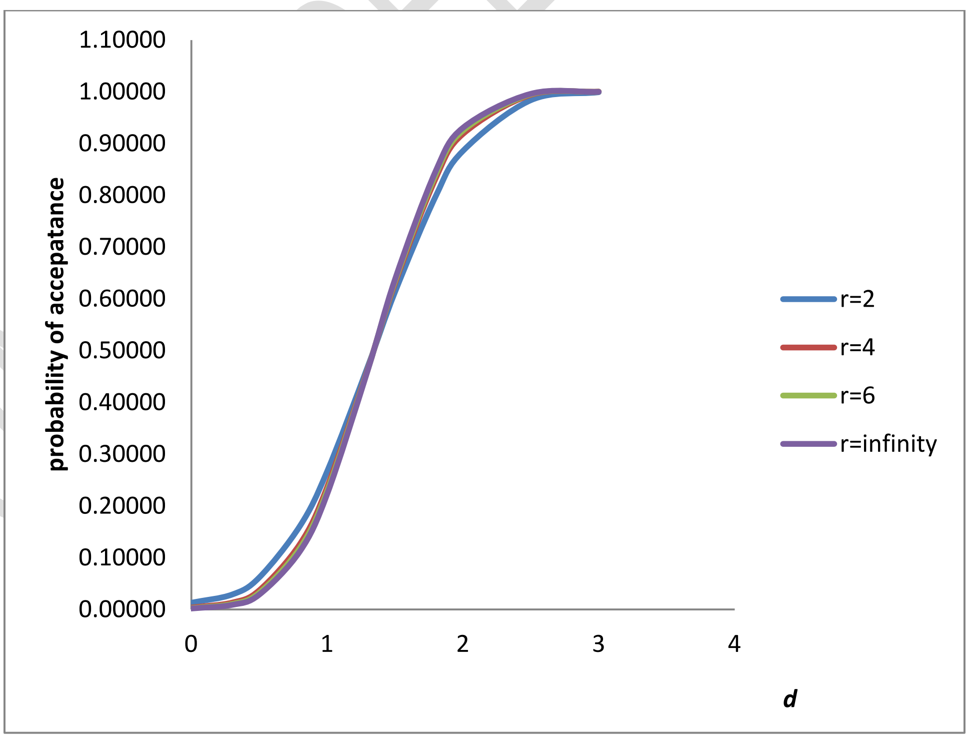

Fig-3: Power curve of average control chart when $(\lambda_3, \lambda_3) = (0, -0.5)$

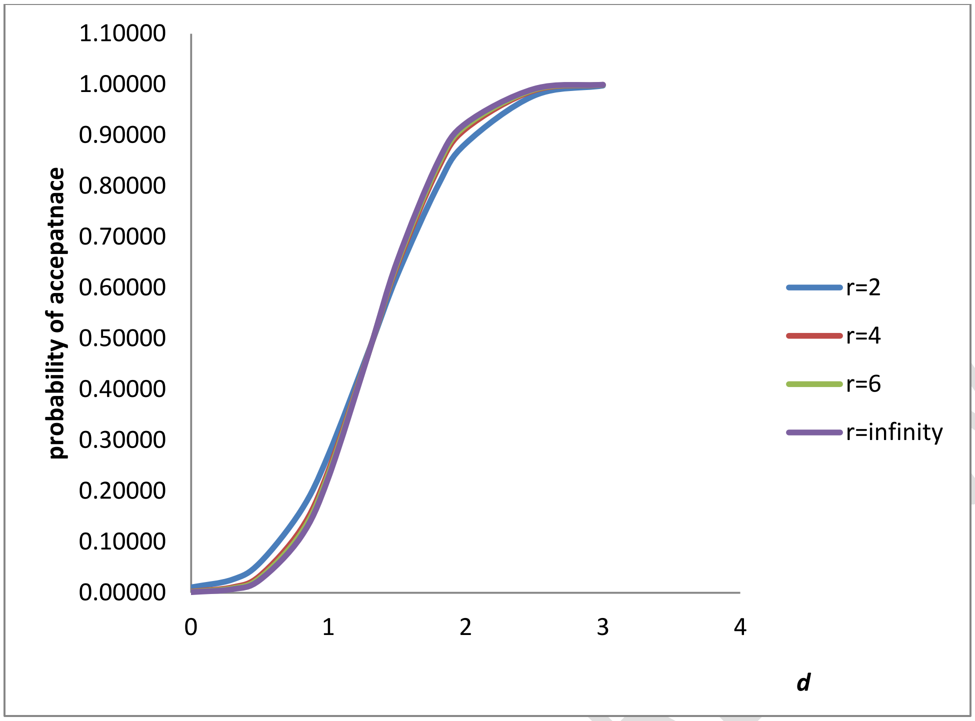

Fig-4: Power curve of average control chart when $(\lambda_3, \lambda_3) = (0.5,0.5)$

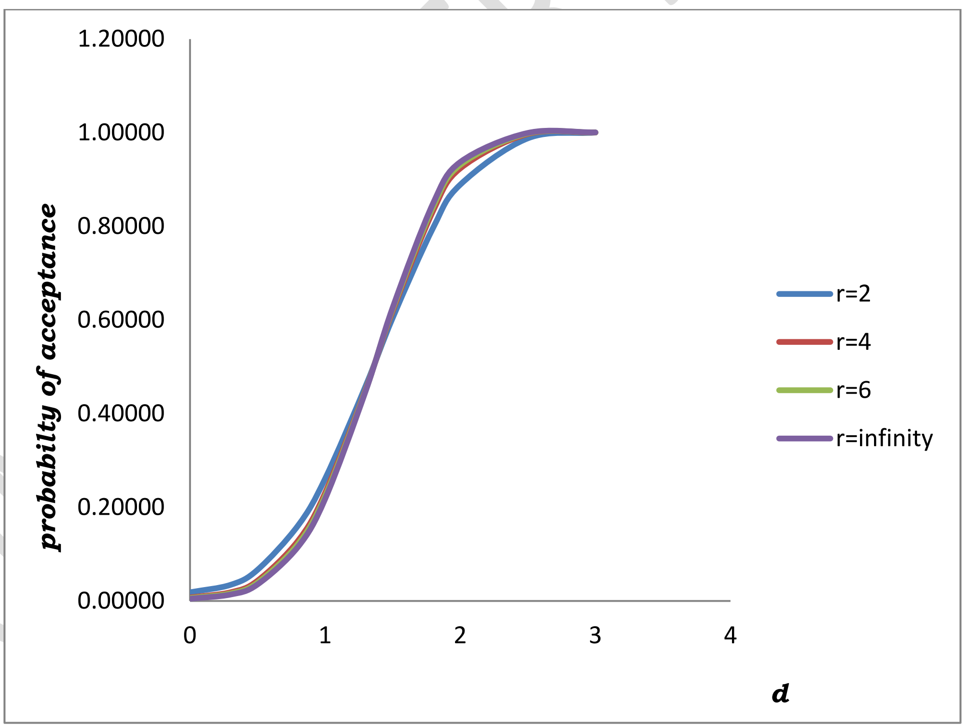

Fig-5: Power curve of average control chart when $(\lambda_3, \lambda_3) = (-0.5, -0.5)$

It is clear from the graphical representation that measurement error seriously affects the power curves for the average control chart under non-normality.

On comparing the values of power function to error free case *viz.* $r = \infty$, with other error rates, it is very clear from the visual presentation that effect of measurement error tends to get closer to error free case when $r = 4$ and $r = 6$ but for $r = 2$ has more serious effect under non-normality, which shows that, in the absence of variability in measurement error, consumers gains undue protection. Since these errors can have considerable effects on the power curves, every effort should be made to reduce the size of errors.

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