Methods for deriving linearly independent solutions of the differential equation with repeated roots

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Abstract

When the eigenvalue λ of a higher order homogeneous linear differential equation with constant coefficients is the repeated root of multiplicity k, the differential equation has exactly k linearly independent solutions. Different textbooks often use different ways to deal with this part of the content. “Advanced Mathematics” by Tongji University and “Ordinary Differential Equations” by Sun Yat-sen University are two commonly used textbooks for science and engineering majors and mathematics majors of universities in China. The former directly gives the conclusion, while the reasoning skills of the latter are not easily understood and mastered by many students, which leading to the degeneration of the mastery of this part into “knowing the conclusion” and “being able to use the conclusion”. This is contrary to the principle that knowledge is only a carrier and teaching must focuses on ability cultivation. We discuss two new methods based on operator decomposition and the solving method for first order linear differential equation. This method is easier to understand and grasp, and can be processed in the same way for real and complex eigenvalues.

Key Words: Repeated roots, ODE, solution.

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1 Introduction

Homogeneous linear n-order differential equations with constant coefficients have exactly n linearly independent special solutions. Different textbooks usually deal with this in different ways. “Ordinary Differential Equations” [1] of Sun Yat-sen University and “Advanced Mathematics” by Tongji University [2] are two textbooks commonly used in mathematics and science and engineering majors of universities in China. “Ordinary Differential Equations” [1] divides the situation into two cases. Firstly, the case of eigenvalue $\lambda = 0$ is discussed, then uses the transformation $u = y(x)e^{\lambda x}$ to change the case of eigenvalue $\lambda \neq 0$ into the former case. The advantage of this method is that the theory can be explained in a smaller space, but it is not good to explain to students how to think of such a treatment?" And the corresponding relationship between the two cases involves the derivation of some relations, which is also difficult for students to understand. "Advanced Mathematics" by Tongji University [2] adopts the way is to guess the "second order constant coefficient linear equation has a solution $e^{\lambda x}$", set another linearly independent solution is $u(x)e^{\lambda x}$, and then transform it into the first order equation about $u(x)$ to find another solution. The advantage of this treatment is it conform to the science and engineering students who focus on the application of relevant conclusions, while the disadvantage is that the reason is not clearly explained. We also refer to a number of relevant textbooks at home and abroad, such as [3, 4, 5], the treatment is basically the same as sun Yat-sen University [1].

At present, both the teaching and learning of professional courses and public basic courses should not only pay attention to the practical needs of reducing class hours and simplifying content, but also accord with the principle that knowledge is only the carrier and teaching focuses on ability cultivation. "Paying attention to ideas and diluting skills" is the eternal pursuit of mathematics courses. For this reason, many teachers and learners constantly discuss some classic contents in the textbook from both teaching and learning aspects [6-9]. The purpose of this paper is to present two new approaches for solving the problem that the differential equation has exactly k linearly independent solutions. These two new processing methods based on the idea of operator decomposition and the solving method for first order linear differential equation. We transform the solving of the high-order problem into the solving of some first-order problems. The interesting point is striving to use the most concise method and thinking to deal with the problem. This method is easier to understand and grasp the ideas and can be solved in the same way for real and complex eigenvalues.

2 The particular solution by the use of the operator decomposition method

Consider the following linear homogeneous differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{2.1}$$

where $a_0, a_1, \cdots, a_{n-1}$ are constants. The characteristic equation of differential equation (2.1) is

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{2.2}$$

According to the fundamental theorem of algebra, the equation (2.2) have exactly n roots in the range of complex numbers, let them be $\lambda_1, \lambda_2, \cdots, \lambda_k$, whose multiplicities are $n_1, n_2, \cdots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$.

Definition 2.1 *For given complex number $r \in \mathbb{C}$, define a first order polynomial operator $D - r$ which operates on the function $y = y(x)$ to produce*

$$(D - r)y = Dy - ry = y' - ry.$$

The important fact about such operators is that any two of them commute:

$$(D - r_1)(D - r_2)y = (D - r_2)(D - r_1)y,$$

for any twice differentiable function $y = y(x)$. So, for $m \in \mathbb{N}$, one has

$$(D - r)^m y = (D - r)(D - r)\cdots(D - r)y.$$

Theorem 2.1 *Suppose the equation* (2.1) *have k different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ of multiplicities $n_1, n_2, \cdots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$. Then the differential equation* (2.1) *have n linear independent solutions.*

Proof. Taking into account the above operator decomposition idea and the case of the characteristic equation (2.2) root, the differential equation (2.1) can be rewritten as

$$(D - \lambda_1)^{n_1}(D - \lambda_2)^{n_2}\cdots(D - \lambda_k)^{n_k} y = 0. \tag{2.3}$$

Clearly, the solution of n_k order equation $(D - \lambda_k)^{n_k} y = 0$ is the solution of the differential equation (2.1). The n_k order differential equation $(D - \lambda_k)^{n_k} y = 0$ can be written as

$$(D - \lambda_k)[(D - \lambda_k)\cdots(D - \lambda_k)y] = 0. \tag{2.4}$$

Let $[(D-\lambda_k)\cdots(D-\lambda_k)y] = y_1$. It is well known that the solution formula of first-order linear differential equation $y'(x) + p(x)y(x) = q(x)$ is

$$y(x) = ce^{-\int p(x)dx} + e^{-\int p(x)dx} \cdot \int q(x)e^{\int p(x)dx}dx.$$

By the use of the above formula, one can get the solution of the first order linear homogeneous differential equation $(D-\lambda_k)y_1 = 0$ is $y_1 = ce^{\lambda_1 x}$. And the solution of the first order linear nonhomogeneous differential equations $(D-\lambda_k)y_2 = y_1$ is $y_2 = (c_1 + c_2 x)e^{\lambda_k x}$. Thus, the solution of differential equation (2.4) can be obtained by analogy

$$y(x) = (c_1 + c_2 x + \cdots + c_{n_1} x^{n_1 - 1})e^{\lambda_k x}.$$

That is to say that when the eigenvalue $\lambda_k \in \mathbb{R}$ is multiplicity n_k, the linearly independent real value special solutions of the n_k order differential equation (2.4) are

$$e^{\lambda_k x}, xe^{\lambda_k x}, \cdots, x^{n_k - 1}e^{\lambda_k x}.$$

A similar method is applied to the n_i equation $(D-\lambda_i)^{n_i} y = 0, (i = 1, 2, \cdots, k-1)$ can conclude the differential equation (2.1) have $n_1 + n_2 + \cdots + n_k = n$ special solutions:

$$\left\{ \begin{array}{l} e^{\lambda_1 x}, xe^{\lambda_1 x}, \cdots, x^{n_1 - 1}e^{\lambda_1 x}; \\ e^{\lambda_2 x}, xe^{\lambda_2 x}, \cdots, x^{n_2 - 1}e^{\lambda_2 x}; \\ \qquad \cdots \\ e^{\lambda_k x}, xe^{\lambda_k x}, \cdots, x^{n_k - 1}e^{\lambda_k x}. \end{array} \right. \tag{2.5}$$

Now consider the case that the characteristic equation have complex repeated roots. Suppose that the repeated eigenvalue $\lambda = \alpha + i\beta$ is multiplicity of k, then $\bar{\lambda} = \alpha - i\beta$ is also a multiplicity of k repeated eigenvalue. Since the above operations are also feasible for complex eigenvalues, we can obtain $2k$ real-valued solutions of the equation (2.1) by using the linear properties of the solutions:

$$\left\{ \begin{array}{l} e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \cdots, x^{k-1}e^{\alpha x}\cos\beta x, \\ e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \cdots, x^{k-1}e^{\alpha x}\sin\beta x. \end{array} \right. \tag{2.6}$$

The proof is complete. ¶

Remark 2.1 *Here, the method of operator decomposition is used to transform the solution of linearly independent solution of higher-order equation into the solution of first-order linear equation that students are familiar with, which is easier to understand and master.*

3 The particular solution by the use of the fundamental solution matrix method

The solution of a higher order differential equation can be transformed into the solution of first-order differential equation system. The conclusion of the special solution of higher order equation when the eigenvalues are repeated roots can also be given directly at the beginning (similar to [2] processing), and then give a more clear explanation in the system of equations part with very simple calculation.

Proof. (Another proof of Theorem 2.1:)

Let $y = y_1, (D - \lambda_k)y_1 = y_2, \cdots, (D - \lambda_k)y_{n_k-1} = y_{n_k}$, then the higher-order equation (2.4) can be written in the form of first-order equation system

$$\begin{cases} y_1' - \lambda_k y_1 = y_2, \\ y_2' - \lambda_k y_2 = y_3, \\ \qquad \vdots \\ y_{n_k-1}' - \lambda_k y_{n_k-1} = y_{n_k}, \\ y_{n_k}' - \lambda_k y_{n_k} = 0. \end{cases} \tag{3.1}$$

Denote

$$y = (y_1, y_2, \cdots, y_{n_k})^T, \qquad A = \begin{pmatrix} \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & \lambda_k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix},$$

then, the differential equation system (3.1) can be written as $y' = Ay$. The matrix A is a Jordan matrix. By the definition of matrix index or the mathematical software Mathematica, simple calculation [3] can obtain the fundamental solution matrix of equations (3.1) is as follows

$$e^{Ax} = \begin{pmatrix} e^{\lambda_1 x} & e^{\lambda_1 x}x & \cdots & \frac{1}{(n_1-2)!}e^{\lambda_1 x}x^{n_1-2} & \frac{1}{(n_1-1)!}e^{\lambda_1 x}x^{n_1-1} \\ 0 & e^{\lambda_1 x} & \cdots & \frac{1}{(n_1-3)!}e^{\lambda_1 x}x^{n_1-3} & \frac{1}{(n_1-2)!}e^{\lambda_1 x}x^{n_1-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_1 x} & e^{\lambda_1 x}x \\ 0 & 0 & \cdots & 0 & e^{\lambda_1 x} \end{pmatrix}. \tag{3.2}$$

According to the relationship between the linear equation system (3.1) and the solutions of the higher order linear equation (2.4), the first row of the matrix exponential (3.2) is n_k linearly independent special solutions of the n_k order equation (2.4) $e^{\lambda_k x}, xe^{\lambda_k x}, \cdots, x^{n_k-1}e^{\lambda_k x}$. Similar method applied to n_i order differential equation $(D - \lambda_i)^{n_i}y = 0, i = 1, 2, \cdots, k-1$, one obtain $n_1 + n_2 + \cdots + n_k = n$ linearly independent special solutions of (2.5). For the case that the characteristic equation has repeated roots, it is easy to deduce the conclusion such as (2.6) from the linear property of the solution. ¶

Remark 3.1 *The textbook [1] only calculate the matrix exponential matrix of the Jordang matrix (Page 127, Example 2). The method given above is convenient for students* *both on mastering the calculation of Jordan exponential matrix and on the other hand, as a direct application, it is also easy to obtain linear independent solution of higher order equations. Some teaching practices show that this kind of treatment is popular with students.*

References

[1] Wang Gaoxiong, Zhou Zhiming, Zhu Siming, Wang Shousong, Ordinary Differential Equations (fourth edition) [M]. Beijing: Higher Education Press, 2020.

[2] Department of Mathematics, Tongji University. Higher Mathematics (7th Edition) [M]. Beijing: Higher Education Press, 2014.

[3] Ding Tongren, Li Chengzhi. Course of Ordinary Differential Equations (2nd edition) [M]. Beijing: Higher Education Press, 2004.

[4] Zhang Weinian, Du Zhengdong, Xu Bing. Ordinary Differential Equations (2nd Edition) [M]. Beijing: Higher Education Press, 2014.

[5] R.K. Miller, A.N. Michel, Ordinary Differential Equations [M]. Academic Press, 1982.

[6] Zhao Linlong, A new method for solving linear equations with constant coefficients [J]. Mathematics in Practice and Theory, 2014, 44(14): 302–308.

[7] Dai Zhonglin, A recursive method for solving linear homogeneous differential equations with constant coefficients [J]. Journal of Sichuan Normal University (Natural Science Edition), 1995, 16(2): 158–160.

[8] Peng Qingying, A new method for solving the basic solution matrix of linear differential equations with constant coefficients [J]. College Mathematics, 2013, 29(6): 120–124.

[9] Feng June, Zhang Qingle, Solution method of homogeneous linear ordinary differential equations based on transition matrix [J]. College Mathematics, 2020, 36(6): 55–62.